Large time behavior of solutions of Hamilton-Jacobi-Bellman equations with superlinear nonlinearity in gradients

Naoyuki Ichihara

(Hiroshima University, Japan)

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HJB equation. $Q := (0, +\infty) \times \mathbb{R}^N$

$$\text{(CP)} \qquad \begin{aligned} \frac{\partial u}{\partial t} - \frac{1}{2}\Delta u + H(x, Du) &= 0 \quad \text{in } Q \\ u &= g \quad \text{on } \partial_p Q \end{aligned}$$

- $(x, p) \mapsto H(x, p)$ is smooth
- $p \mapsto H(x, p)$ is convex and superlinear
- $g \in C_p(\mathbb{R}^N)$, $\inf_{\mathbb{R}^N} g > -\infty$ bounded below

Objective. Long-time behavior of the solution:

$$\frac{u(t,x)}{t} \longrightarrow ? \qquad u(t,x) \longrightarrow ? \qquad (t \to \infty)$$

Related results.

PDE: Barles-Souganidis('01), Souplet-Zhang('06), Barles-Porretta-Tchamba('10), Fujita-Ishii-Loreti('06), etc.

Probability (math finance): Fleming-Sheu('99), Nagai('03,'10), Hata-Nagai-Sheu('10), I.-Sheu('10), etc.

Example 1 (quadratic case).

$$H(x,p) = \frac{1}{2}a(x)p \cdot p + b(x) \cdot p - V(x)$$

- $\alpha_1 I \leq a(x) \leq \alpha_2 I, \qquad \alpha_1, \alpha_2 > 0$
- $\beta_1 |x|^2 - C \leq b(x) \cdot x \leq \beta_2 |x|^2 + C, \quad \beta_1, \beta_2 \in \mathbb{R}, \quad C > 0$
- $\gamma_1 |x|^2 - C \leq V(x) \leq \gamma_2 |x|^2 + C, \quad \gamma_1, \gamma_2 \in \mathbb{R}, \quad C > 0$
- Either $\gamma_1 > 0$ or $\beta_1 > 0\,, \quad \gamma_1 > -\beta_1^2/\alpha_2$

Example 2 (superlinear). $H(x,p) = \frac{1}{m}|p|^m - f(x)$

$$\text{(CP)} \qquad \begin{aligned} \frac{\partial u}{\partial t} - \frac{1}{2}\Delta u + \frac{1}{m}|Du|^m &= f \quad \text{in } Q \\ u &= g \quad \text{on } \partial_p Q \end{aligned}$$

- $m > 1$
- $C^{-1}|x|^{\beta} - C \leq f(x) \leq C(1 + |x|^{\beta}), \quad \beta > 0,\ C > 0$

In this talk, we discuss Example 2 only. (for simplicity)

Stochastic control interpretation

$$u(T,x) = \inf_{\xi} E\left[\int_0^T \left\{\frac{1}{m^*}|\xi_t|^{m^*} + f(X_t^{\xi})\right\} dt + g(X_T^{\xi})\right]$$

$$X_t^{\xi} = x - \int_0^t \xi_s \, dt + W_t, \qquad 0 \le t \le T$$

- $m^* := \dfrac{m}{m-1} > 1$
- $W = (W_t)$: standard $(\mathcal{F}_t)$-Brownian motion in $\mathbb{R}^N$
- $\xi = (\xi_t)$: $\mathbb{R}^N$-valued, $(\mathcal{F}_t)$-prog. m'ble (control)

Theorem 1. *(CP) has a minimal solution u in the class*

$$\Phi := \{u \in C^{1,2}(Q) \cap C_p(\overline{Q}) \mid \inf_{[0,T]\times\mathbb{R}^N} u > -\infty, \ \ \forall T > 0\}.$$

Moreover, u coincides with the value function of the stochastic control problem above.

Remark. Suppose $m \geq 2$ or $g(x) \leq C(1 + |x|^{(\beta/m)+1})$. Then, the solution is unique in Φ.

Long-time behavior. $u(T,x) \approx \lambda^* T + \varphi(x) + c \quad (T \gg 1)$

$$\frac{u(T,\cdot)}{T} \longrightarrow \lambda^* \qquad \text{in } C(\mathbb{R}^N) \text{ as } T \to \infty$$

$$u(T,\cdot) - \lambda^* T \longrightarrow \varphi(\cdot) + c \qquad \text{in } C(\mathbb{R}^N) \text{ as } T \to \infty$$

Goal. Characterization of (λ^*, φ) and c.

Remark. (λ^*, φ) does not depend on g, while c does.

Simplest example. $N = 1, \quad m = 2, \quad f(x) = x^2, \quad g = 0$

$$\frac{\partial u}{\partial t} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = x^2, \qquad u\big|_{t=0} = 0$$

$$u(T,x) = \sqrt{2}\,t + \frac{e^{2\sqrt{2}T} - 1}{e^{2\sqrt{2}T} + 1}\,\frac{\sqrt{2}}{2}\,x^2 - \log\frac{e^{2\sqrt{2}T}}{e^{2\sqrt{2}T} + 1} - \log 2$$

$$\approx \sqrt{2}\,T + \frac{\sqrt{2}}{2}\,x^2 - \log 2 \qquad (T \gg 1)$$

Limiting equation. (λ^*, φ) is a solution of

$$\text{(EP)} \qquad \lambda - \frac{1}{2}\Delta\phi + \frac{1}{m}|D\phi|^m = f \quad \text{in } \mathbb{R}^N, \quad \phi(0) = 0$$

We call it an ergodic problem.

Remark. (EP) has many solutions (λ, ϕ).

How to select the correct candidate among them ?

Simplest example. $N = 1, \quad m = 2, \quad f(x) = x^2$

$$(*) \qquad \lambda - \frac{1}{2}\phi_{xx} + \frac{1}{2}\phi_x^2 = x^2, \quad \phi(0) = 0$$

$(\lambda_1, \phi_1) = (\sqrt{2}, \dfrac{\sqrt{2}}{2}x^2)$ is a solution of $(*)$.

$(\lambda_2, \phi_2) = (-\sqrt{2}, -\dfrac{\sqrt{2}}{2}x^2)$ is also a solution of $(*)$.

Remark. The former turns out to be the correct one.

Characterization of the candidate.

$$\text{(EP)} \qquad \lambda - \frac{1}{2}\Delta\phi + \frac{1}{m}|D\phi|^m = f \quad \text{in } \mathbb{R}^N, \quad \phi(0) = 0$$

Theorem 2. *(a) There exists a constant* $\lambda^* \in \mathbb{R}$ *such that*

$$(EP)_\lambda \text{ has a solution } \phi \quad \Longleftrightarrow \quad \lambda \le \lambda^*$$

(b) Let (λ, ϕ) *be any solution of (EP). Then,*

$$\lambda = \lambda^* \quad \text{iff} \quad \inf_{\mathbb{R}^N} \phi > -\infty$$

Moreover, such solution is unique.

Stochastic control interpretation.

Let (λ^*, φ) be the unique solution of (EP).

$$\lambda^* = \inf_{\xi} \liminf_{T\to\infty} \frac{1}{T} E\left[\int_0^T \Big(\frac{1}{m^*}|\xi_t|^{m^*} + f(X_t^{\xi})\Big)\, dt\right]$$

$$X_t^{\xi} = x - \int_0^t \xi_s\, dt + W_t$$

Optimal feedback process. $\xi^*(x) := |D\varphi(x)|^{m-2} D\varphi(x)$

$$dX_t = -\xi^*(X_t)\, dt + dW_t, \qquad t \geq 0$$

Remark. $X = (X_t)$ is ergodic (positive recurrent).

Simplest example. $N = 1, \quad m = 2, \quad f(x) = x^2$

$$\lambda^* - \frac{1}{2}\varphi_{xx} + \frac{1}{2}\varphi_x^2 = x^2, \quad \varphi(0) = 0$$

$\lambda^* = \sqrt{2}, \quad \varphi(x) = \dfrac{\sqrt{2}}{2}x^2$

$\xi^*(x) = |D\varphi(x)|^{m-2}D\varphi(x) = \varphi_x(x) = \sqrt{2}x$

$dX_t = -\sqrt{2}X_t\,dt + dW_t$ (Ornstein-Uhlenbeck process)

Invariant measure: $\mu(x) = Z^{-1}\exp(-\sqrt{2}x^2)$ (Gaussian)

Main results. (λ^*, φ): unique solution of (EP)

Theorem 3. *For any solution* $u \in \Phi$ *of* (CP),

$$\frac{u(T,\cdot)}{T} \longrightarrow \lambda^* \quad \textit{in} \ \ C(\mathbb{R}^N) \quad \textit{as} \ \ T \to \infty.$$

Theorem 4. *Suppose that* $\beta \geq m^* = \dfrac{m}{m-1}$. *Then,*

$$u(T,\cdot) - (\lambda^* T + \varphi(\cdot)) \longrightarrow \exists c \quad \textit{in} \ \ C(\mathbb{R}^N) \quad \textit{as} \ \ T \to \infty.$$

Remark. We do not know if $\beta \geq m^*$ is removable or not.

Thank you for your attention!